Analysis of Fractional Nonlinear Differential Equations Using the Homotopy Perturbation Method

Mehmet Ali Balcı and Ahmet Yıldırım

Ege University, Department of Mathematics, 35100 Bornova-İzmir, Turkey

Reprint requests to M. A. B.; E-mail: mehmetalibalci.ege@gmail.com

Z. Naturforsch. **66a,** 87 – 92 (2011); received March 22, 2010 / revised July 8, 2010

In this study, we used the homotopy perturbation method (HPM) for solving fractional nonlinear differential equations. Three models with fractional-time derivative of order α, $0 < \alpha < 1$, are considered and solved. The numerical results demonstrate that this method is relatively accurate and easily implemented.

Key words: Homotopy Perturbation Method; Time Fractional Nonlinear Fractional Differential Equations.

1. Introduction

In recent years, it has been found that derivatives of non-integer order are very effective for the description of many physical phenomena such as rheology, damping laws, and diffusion processes. These findings invoked the growing interest on studies of the fractal calculus in various fields such as physics, chemistry, and engineering [1 – 4]. In general, there exists no method that yields an exact solution for a fractional differential equation. Only approximate solutions can be derived using the linearization or perturbation methods. Some authors applied the homotopy perturbation method (HPM) [5 – 9], the variational iteration method (VIM) [10 – 12], and the reduced differential transform method [13] to fractional differential equations and revealed that HPM and VIM are alternative analytical methods for solving such type equations. Nobel Laureate Gerardus't Hooft once remarked that discrete space-time is the most radical and logical viewpoint of reality. Physical phenomena in a fractal space-time are describable by the fractional calculus [14]. The fractional equations are used to describe discontinuous problems. According to the fractal space-time theory (El Naschie's e-infinity theory), time and space are discontinuous, and the fractional model is the best candidate to describe such problems [14].

In this paper, we will use HPM for solving time fractional nonlinear fractional differentials. This paper gives an important example of the fractional Korteweg-de Vries (KdV) equation, which, according to a recent report, admits a fractional variational principle [15]. Recently, El-Wakil et al. [16] used the Adomian decomposition method for solving the governing problem.

The homotopy perturbation method was proposed by the Chinese mathematician Ji-Huan He [17 – 19]. This technique has been employed to solve a large variety of linear and nonlinear problems [5 – 7, 20 – 22]. Unlike classical techniques, the nonlinear equations are solved easily and elegantly without transforming the equation by using the HPM. The technique has many advantages over the classical techniques, mainly, it avoids linearization and perturbation in order to find explicit solutions of a given nonlinear equations.

2. Definitions

Definition 2.1: A real function $f(x)$, $x > 0$, is said to be in the space C_M, $M \in \mathbb{R}$, if there exists a real number $p(> M)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_M^m if $f^m \in C_M$, $m \in \mathbb{N}$.

Definition 2.2: If $f(x) \in C[a,b]$ and $a < x < b$, then

$$\mathrm{I}_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \mathrm{d}t,$$

where $-\infty < \alpha < \infty$, is called the Riemann-Liouville fractional integral operator of order α.

Definition 2.3: For $0 < \alpha < 1$, we let

$$\mathrm{D}_a^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(t)}{(x-t)^\alpha} \mathrm{d}t,$$

which is called the Riemann-Liouville fractional derivative operator of order α.

Lemma 2.1: If $f(x)$ is an absolutely continuous function in $[a,b]$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{I}^\alpha f(x) = \mathrm{I}^\alpha \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{x^{\alpha-1}}{\Gamma(\alpha)} f(0).$$

Lemma 2.2: If $f(x)$ is an absolutely continuous function in $[a,b]$, $f''(x)$ exists and $f'(0) = 0$, then

$$\mathrm{D}^\alpha \mathrm{D}^\beta f(x) = \mathrm{D}^{\alpha+\beta} f(x),$$

where $\alpha + \beta \in (1,2)$.

3. Illustrative Examples

Example 1: Firstly, we consider the nonlinear fractional KdV equation [16] $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + (p+1)u^p u_x + u_{xxxx} = 0$, $t > 0$, $0 < \alpha < 1$, with the initial condition $u(x,0) = A[\mathrm{sech}^2(Kx)]^{\frac{1}{p}}$, where $p > 0$, A and K are constants. $\frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional time derivative operator of order α. This equation has a wide range of applications in plasma physics, fluid physics, capillary-gravity waves, nonlinear optics, and chemical physics.

We construct the homotopy

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \bar{p}\left[(p+1)u^p u_x + u_{xxxx}\right] = 0, \quad \bar{p} \in [0,1]. \quad (1)$$

In view of HPM, we use the homotopy parameter p to expand the solution

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots \quad (2)$$

Substituting (2) into (1) and equating the coefficients of like powers of p, we get following set of differential equations:

$$\bar{p}^0 : \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0,$$

$$\bar{p}^1 : \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} + (p+1)u_0^p (u_0)_x + (u_0)_{xxxx} = 0,$$

$$\bar{p}^2 : \frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} + (u_0^p u_1)_x + (u_1)_{xxxx} = 0,$$

$$\bar{p}^3 : \frac{\partial^\alpha u_3(x,t)}{\partial t^\alpha} + (pu_0^{p-1} u_1^2/2 + u_0^p u_2)_x + (u_2)_{xxxx} = 0,$$

$\ldots$

The solution reads

$$u_0 = 0.2[\mathrm{sech}(0.1x)]^{\frac{1}{4}},$$

$$u_1(x,t) = -0.136 \cdot 10^{-5}\Big[\cosh(0.1x)^4 - 78\cosh(0.1x)^2 + 105 - 80\cosh(0.1x)\sinh(0.1x)t^{3/4}\Big] \Big/ \left[\cosh(0.1x)^6 \left(\frac{1}{\cosh(0.1x)^2}\right)^{3/4}\right],$$

$$u_2(x,t) = -0.75 \cdot 10^{-16} t^{9/4} \Big[0.24 \cdot 10^{12} \cosh(0.1x)^2 - 0.17 \cdot 10^{12} - 0.87 \cdot 10^{11} \cosh(0.1x)^4 + 0.68 \cdot 10^{11} \cosh(0.1x)\sinh(0.1x) - 85005 \cdot \cosh(0.1x)^8 + 0.41 \cdot 10^{10} \cosh(0.1x)^6 - 0.49 \cdot 10^{11} \cosh(0.1x)^3 \sinh(0.1x) + 0.42 \cdot 10^{10} \cosh(0.1x)^5 \sinh(0.1x)\Big] \Big/ \left[\cosh(0.1x)^{10} \left(\frac{1}{\cosh(0.1x)^2}\right)^{3/4}\right],$$

and so on. In the same manner the rest of components of the homotopy perturbation series of $u(x,t)$ can be obtained. Then the solution of the homotopy method may be constructed explicitly. The behaviour of the homotopy solution for the fractional KdV equation with different values of the fractional time derivative order α are shown graphically in Figure 1.

In case of $\alpha = 1$, the homotopy solution is compared with the exact solution of KdV

$$u(x,t) = A[\mathrm{sech}^2(Kx - ct)]^{\frac{1}{p}},$$

where c is a constant. As shown in Figure 2 the two solutions are equivalent.

Example 2: The second example is the fractional time derivative Burgers-Fisher equation [16]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + pu^r u_x - u_{xx} = qu(1-u^r), \quad t > 0, \quad 0 < \alpha < 1, \quad (3)$$

with the initial condition

$$u(x,0) = \sqrt{1/2 - \tanh(Kx)/2}, \quad (4)$$

where p and q are constants and $K = 1/2(1+r)$.

This equation has been used as a basis for a wide variety of models for the spatial of gene in population and chemical wave propagation.

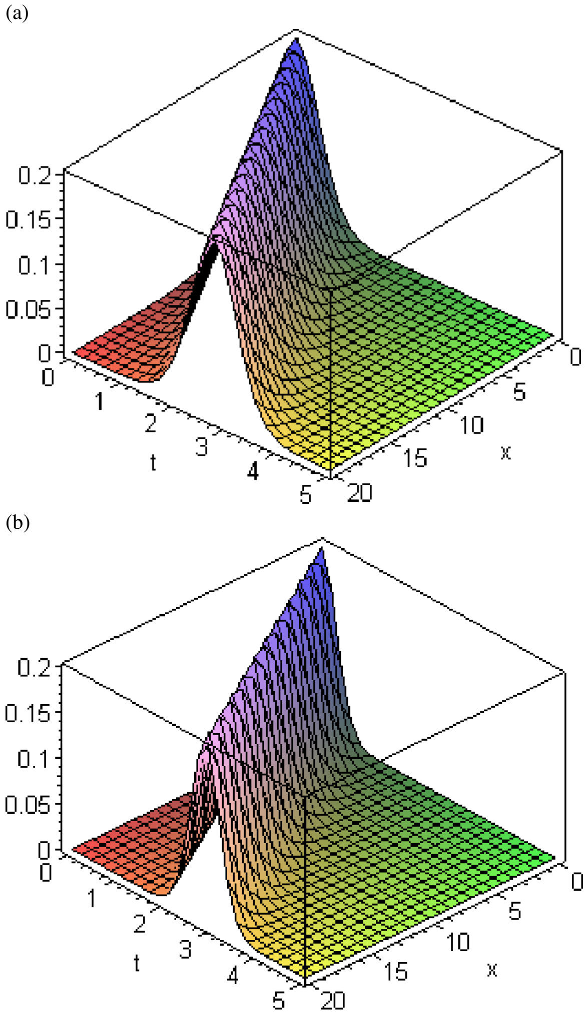


Fig. 1 (colour online). HPM solution of the generalized fractional time KdV equation with fixed values $p = 6$, $K = 0.1$, $A = 0.2$, and $c = 1$: (a) $\alpha = 1/2$, (b) $\alpha = 3/4$.

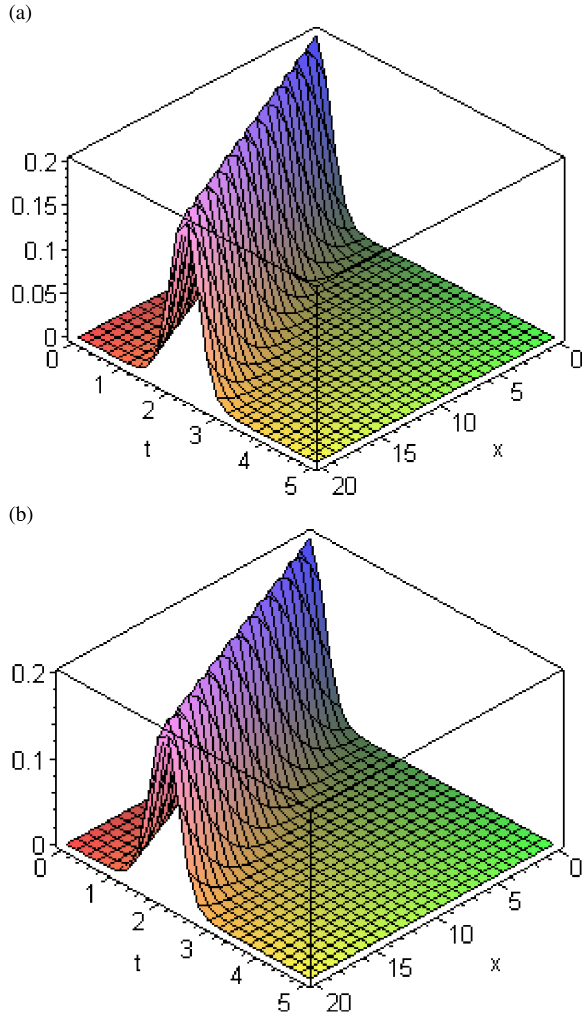


Fig. 2 (colour online). Solution of the generalized fractional time KdV equation with fixed values $p = 6$, $K = 0.1$, $A = 0.2$, $c = 1$, and $\alpha = 1$: (a) HPM solution, (b) exact solution.

We construct the homotopy which satisfies the relation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \bar{p}[pu^r u_x - u_{xx} - qu(1-u^r)] = 0, \quad \bar{p} \in [0,1]. \tag{5}$$

Substituting (2) into (5) and equating the coefficients of like powers of p, we get following set of differential equations:

$$\bar{p}^0 : \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0,$$
$$\bar{p}^1 : \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} + pu_0^r(u_0)_x - (u_0)_{xx} - qu_0(1-u_0)^r = 0,$$
$$\bar{p}^2 : \frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} + pu_0^r(u_1)_x - (u_1)_{xx} - q(r+1)u_0^r u_1 = 0,$$
$$\ldots$$

In the same manner (as Example 1) the rest of the components can be obtained.

The numerical behaviour of the approximate solutions of the homotopy method with different values of fractional time derivative order α are shown graphically in Figure 3. From this figure we see that our solution agrees with the exact solution of the Burgers-Fisher equation

$$u(x,t) = [1/2 - \tanh(K(x-ct))/2]^{1/r}$$

in case of $\alpha = 1$ and $c = \frac{p^2+q(1+r)^2}{p(1+r)}$.

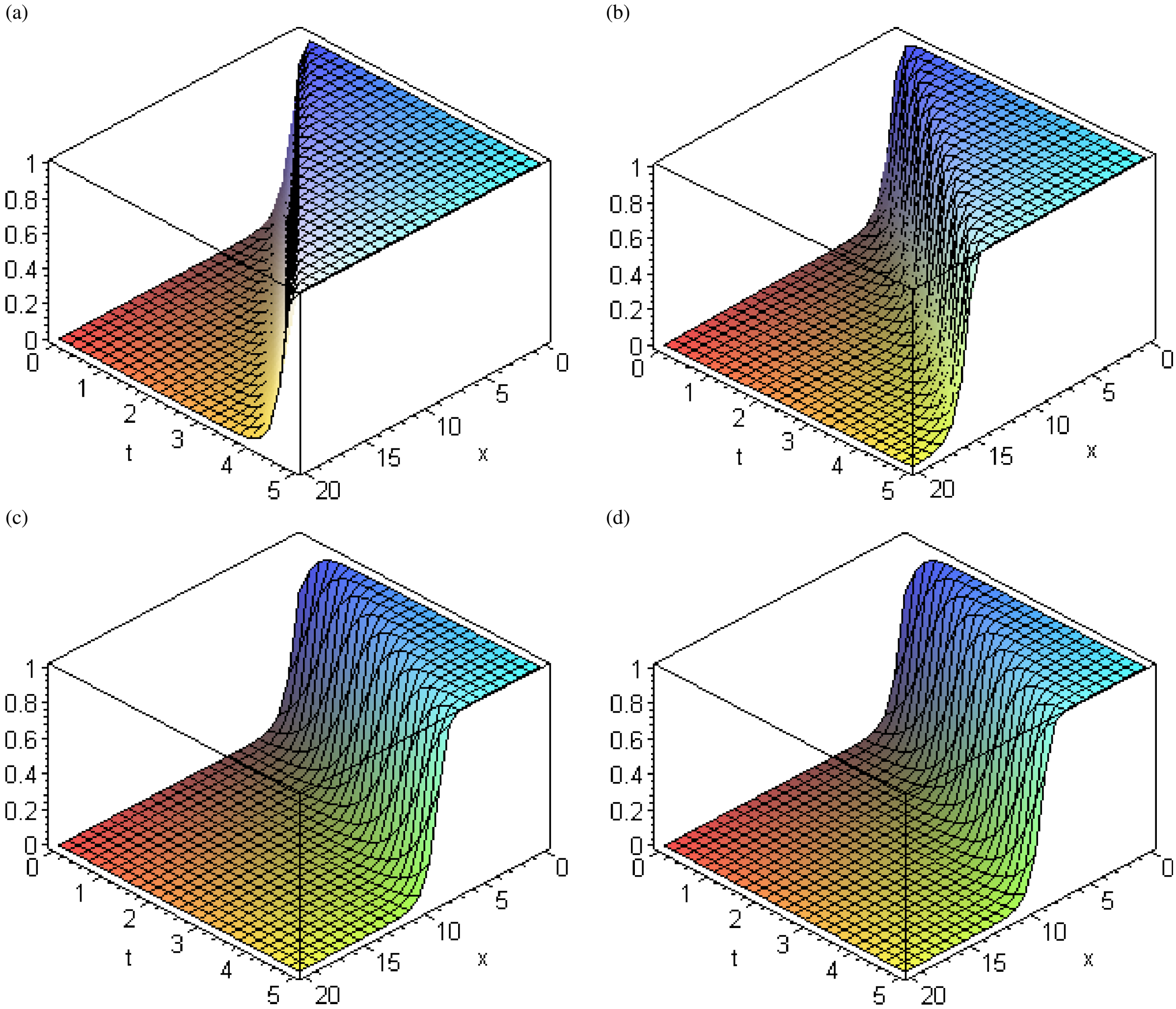


Fig. 3 (colour online). HPM solution of the generalized fractional Burgers-Fisher equation with fixed values $p = 0.1$, $r = 4$, and $q = -0.0025$: (a) $\alpha = 1/2$, (b) $\alpha = 3/4$. Solution of the generalized fractional Burgers-Fisher equation with fixed values $p = 0.1$, $r = 2$, and $q = -0.0025$: (c) HPM solution, (d) Exact solution.

Example 3: Finally, we consider the time fractional coupled system of the diffusion-reaction equation [16]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = u(1-u^2-v) + u_{xx}, \quad t > 0, \quad 0 < \alpha < 1, \quad \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = v(1-u-v) + v_{xx} \tag{6}$$

with initial conditions

$$u(x,0) = \frac{\mathrm{e}^{kx}}{(1+\mathrm{e}^{kx})}, \quad v(x,0) = \frac{1+(3/4)\mathrm{e}^{kx}}{[1+\mathrm{e}^{kx}]^2}, \tag{7}$$

where k is constant.

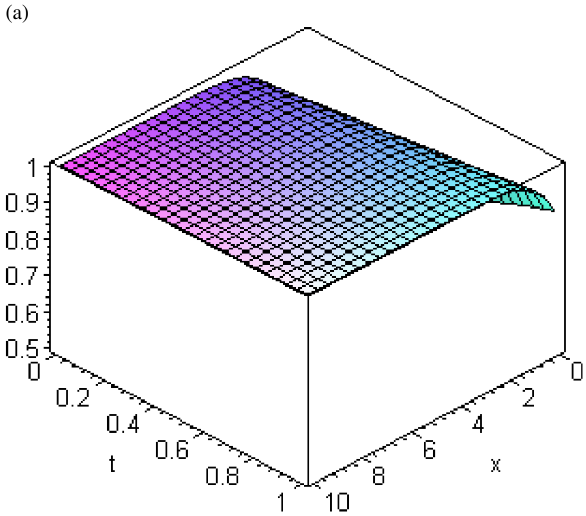
We construct the homotopy which satisfies the relation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \bar{p}[u(u^2+v-1) - u_{xx}] = 0, \quad \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \bar{p}[v(u+v-1) - v_{xx}] = 0, \quad \bar{p} \in [0,1]. \tag{8}$$

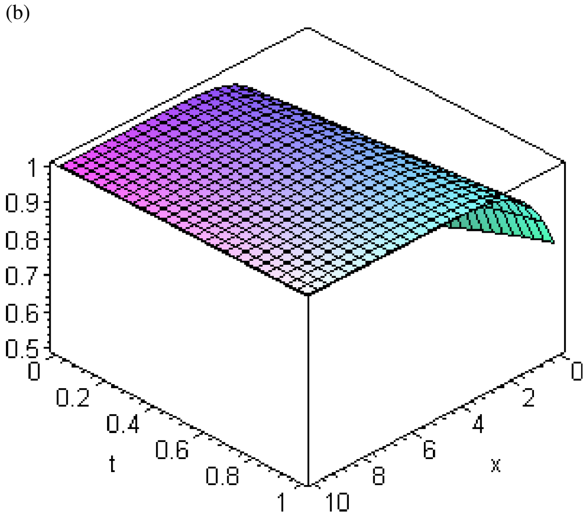
Substituting (2) into (8) and equating the coefficients of like powers of p, we get following set of differential equations:

$$\bar{p}^0 : \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0, \quad \frac{\partial^\alpha v_0(x,t)}{\partial t^\alpha} = 0,$$

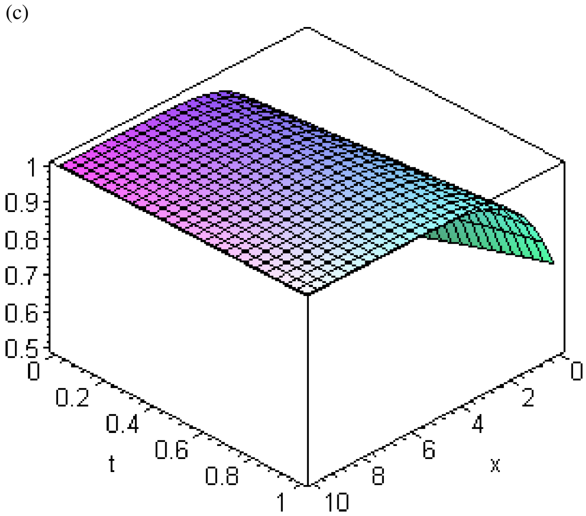
(a)



(b)



(c)



← Fig. 4 (colour online). HPM solution of (6) (for u) with fixed values $k = 1$ and $c = 1$: (a) $\alpha = 1/2$, (b) $\alpha = 3/4$, (c) $\alpha = 1$.

$$\bar{p}^1 : \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} + u_0^3 + u_0 v_0 - u_0 - (u_0)_{xx} = 0,$$
$$\frac{\partial^\alpha v_1(x,t)}{\partial t^\alpha} + u_0 v_0 + v_0^2 - v_0 - (v_0)_{xx} = 0,$$
$$\bar{p}^2 : \frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} + 3u_0^2 u_1 + u_1 v_0 + u_0 v_1 - u_1 - (u_1)_{xx} = 0,$$
$$\frac{\partial^\alpha v_2(x,t)}{\partial t^\alpha} + u_1 v_0 + u_0 v_1 + 2v_0 v_1 - v_1 - (v_1)_{xx} = 0,$$
$$\ldots$$

The solution reads

$$u_0 = \frac{\mathrm{e}^{kx}}{(1+\mathrm{e}^{kx})}, \quad v_0 = \frac{1+\frac{3}{4}\mathrm{e}^{kx}}{[1+\mathrm{e}^{kx}]^2},$$
$$u_1(x,t) := -\frac{\mathrm{e}^{kx}(-5\mathrm{e}^{kx} - 4k^2 + 4k^2\mathrm{e}^{kx})t^\alpha}{4(1+\mathrm{e}^{kx})^3\Gamma(\alpha+1)},$$
$$v_1 := \frac{\mathrm{e}^{kx}(4 + 3\mathrm{e}^{kx} - 20k^2 + 16k^2\mathrm{e}^{kx} + 12k^2\mathrm{e}^{2kx})t^\alpha}{16(1+\mathrm{e}^{kx})^4\Gamma(\alpha+1)},$$
$$u_2(x,t) := -\frac{1}{8}\mathrm{e}^{kx}t^{3\alpha}\Big[2\mathrm{e}^{kx} - 8k^4 + 88k^4\mathrm{e}^{kx} - 11\mathrm{e}^{(2kx)} - 60k^2\mathrm{e}^{kx} + 20\mathrm{e}^{(3kx)} - 20k^2\mathrm{e}^{3kx} + 104k^2\mathrm{e}^{2kx} - 88k^4\mathrm{e}^{2kx} + 8k^4\mathrm{e}^{3kx}\Big]\Big/\Big[(1+\mathrm{e}^{kx})^5\Gamma(\alpha+1)\Gamma(2\alpha+1)\Big],$$
$$v_2(x,t) := -\frac{1}{32}\mathrm{e}^{kx}t^{3\alpha}\Big[-528k^4\mathrm{e}^{2kx} - 50\mathrm{e}^{kx} - 8k^2\mathrm{e}^{2kx} + 24k^4\mathrm{e}^{4kx} - 112k^4\mathrm{e}^{3kx} + 36k^2\mathrm{e}^{3kx} - 40k^4 - 8 - 30\mathrm{e}^{3kx} - 73\mathrm{e}^{2kx} - 92k^2\mathrm{e}^{kx} + 528k^4\mathrm{e}^{kx} + 16k^2\Big]\Big/ \Big[(1+\mathrm{e}^{kx})^6\Gamma(\alpha+1)\Gamma(2\alpha+1)\Big].$$

The numerical behaviour of approximate solutions of homotopy method with different values of fractional time derivative order α are shown graphically in Figure 4.

It is noted that in the case $\alpha = 1$, the homotopy solution is equivalent to the exact solution

$$u(z) = \frac{\mathrm{e}^{kz}}{(1+\mathrm{e}^{kz})}, \quad v(z) = \frac{1+(3/4)\mathrm{e}^{kz}}{(1+\mathrm{e}^{kz})^2},$$

where $z = x + ct$.

4. Conclusion

In this study, we used the homotopy perturbation method for solving fractional time derivative nonlinear partial differential equations. We showed the usefulness of the homotopy perturbation method by three examples. It is clear that HPM avoids linearization and unrealistic assumptions and provides an efficient numerical solution. The results so obtained reinforce the conclusions made by many researchers that the efficiency of the homotopy perturbation method and related phenomena gives it much wider applicability.

Acknowledgement

Authors sincerely thank the unknown reviewers for their constructive comments and suggestions.

[1] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego 1999.
[2] R. L. Bagley and P. J. Torvik, J. Appl. Mech. **51**, 294 (1984).
[3] M. Caputo, Elasticita' e Dissipazione, Zanichelli, Bologna, 1969.
[4] L. E. Suarez and A. Shokooh, ASME J. Appl. Mech. **64**, 629 (1997).
[5] S. Momani and Z. Odibat, Phys. Lett. A **365**, 345 (2007).
[6] Z. Z. Ganji and D. D. Ganji, H. Jafari, and M. Rostamian, Topol. Meth. Nonlinear Anal. **31**, 341 (2008).
[7] A. Yıldırım, Int. J. Nonlinear Sci. Numer. Simul. **10**, 445 (2009).
[8] M. M. Mousa and A. Kaltayev, Z. Naturforsch. **64a**, 788 (2009).
[9] N. A. Khan, A. Ara, S. A. Ali, and A. Mahmood, Int. J. Nonlinear Sci. Numer. Simul. **10**, 1127 (2009).
[10] S. Momani, Z. Odibat, and I. Hashim, Topol. Meth. Nonlinear Anal. **31**, 211 (2008).
[11] Z. Odibat and S. Momani, Topol. Meth. Nonlinear Anal. **31**, 227 (2008).
[12] I. Ates and A. Yildirim, Int. J. Nonlinear Sci. Numer. Simul. **10**, 877 (2009).
[13] Y. Keskin and G. Oturanc, Nonlinear Sci. Lett. A **1**, 207 (2010).
[14] J.-H. He, G.-C. Wu, and F. Austin, Nonlinear Sci. Lett. A **1**, 1 (2010).
[15] G. C. Wu and J.-H. He, Nonlinear Sci. Lett. A **1**, 281 (2010).
[16] S. A. El-Wakil, A. Elhanbaly, and M. A. Abdou, Appl. Math. Comput. **182**, 313 (2006).
[17] J.-H. He, Int. J. Nonlinear Sci. Numer. Simul. **6**, 207 (2005).
[18] J.-H. He, Appl. Math. Comput. **151**, 287 (2004).
[19] J.-H. He, Int. J. Nonlinear Mech. **35**, 37 (2000).
[20] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, Int. J Nonlinear Sci. Numer. Simul. **10**, 227 (2009).
[21] S. T. Mohyud-Din and M. A. Noor, Appl. Appl. Math. **3**, 224 (2008).
[22] M. A. Noor and S. T. Mohyud-Din, Int. J. Nonlinear Sci. Numer. Simul. **9** 395 (2008).